Math 210-1, Fall 2009
Instructor : F. Brock
Final Exam, February 4, 50 points

1. Evaluate the following limits, if they exist. If they do not exist, say so. Justify your answer (5 points each):

(a) $\lim_{x\to\infty} (a^x + b^x)^{1/x}$, $(a > 1, b > 1)$.

(b) $\lim_{x\to 0} \dfrac{1 - \cos x \cosh x}{x^4}$.

(c) $\lim_{x\to+\infty} \left(x - x^2 \ln\left[1 + \sin\dfrac{1}{x}\right]\right)$.

2.(a) (5 points) Evaluate

$$\int_0^1 \frac{\sin x}{x}\, dx$$

with an error less than $2 \cdot 10^{-3}$.
(b) (5 points) Show that

$$\tan x \geq x + \frac{x^3}{3} \quad \text{for all } x \in [0, \tfrac{\pi}{2}) \ .$$

(c) (5 points) Let $a > 0$, and define $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) := |x|^a \cos\frac{1}{x}$ if $x \neq 0$, and $f_a(0) := 0$. For which values of a,
(i) f_a is differentiable ,
(ii) f_a' exists and is bounded,
(iii) f_a' exists and is continuous ?
Justify your answers !

3.(a) (5 points) Let $f_n(x) := n^2x^2(1-x)^n$, $(n \in \mathbb{N},\ x \in [0,1])$. Evaluate the limit function $f(x) = \lim_{n\to\infty} f_n(x)$. On which sets converges the sequence uniformly ? Justify your answer !
(b) (5 points) For which x converges the series

$$\sum_{n=1}^{\infty} \ln \sqrt[n]{1 + \frac{x^2}{n}} \quad ?$$

On which sets converges the series uniformly ?

4.(a) (5 points) Does the improper integral

$$\int_2^{\infty} \frac{\sin x}{\ln x}\, dx$$

converge ? Justify your answer !
(b) (5 points) For which values of p and q converges the improper integral

$$\int_1^{+\infty} x^q (\ln x)^p\, dx \quad ?$$

Justify your answer !